Index-Free Multiagent Systems:
An Eulerian Approach [*]

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Abstract: Since the properties of systems comprising many homogeneous agents may be expected to be independent of how the agents happen to be indexed, it should be possible to formulate and solve multiagent control problems in an index-free way. In this paper we provide such an approach, based on an indicator distribution representation, which results in integro-differential dynamics that parallel and extend those obtained within the traditional indexed formulations. Conservation and stability properties are proven; a compatible geometric structure is constructed for the indexed representation; and a discrete analogue is presented which illustrates that for certain problems the Eulerian viewpoint results in very simple controllers.

1. INTRODUCTION

In much of the work to date in the multiagent controls literature it is assumed that there is an indexed collection of agents with states $x_1, \cdots, x_N \in \mathbb{R}^n$. Then, these individual states are collected in a single vector $x \in \mathbb{R}^{nN}$, and analysis proceeds from this point. Examples include Olfati-Saber and Murray (2004), Xiao and Boyd (2003), Egerstedt and Hu (2001), and Tan and Lewis (1996).

For systems in which there are many homogeneous agents, it may be argued that this is an unnatural approach because it requires that agents be indexed, whereas the fundamental properties of such systems should be independent of agent labeling. One way of dealing with this is to introduce an equivalence relation between states, defined

$$x \sim y \iff \exists P \in \pi(N) \text{ s.t. } x = (P \otimes I)y \qquad (1)$$

for any $x, y \in \mathbb{R}^{nN}$, where $\pi(N)$ denotes the group of $N \times N$ permutation matrices and $\otimes$ denotes the Kronecker product. Essentially, one is then looking for properties of the system that are invariant under permutation of the agent indices or for methods by which the agents can agree on a permutation; this is the approach taken in e.g. Twu and Egerstedt (2010) and Zavlanos and Pappas (2007).

In this paper, we propose an alternative formulation which avoids the need for an equivalence relation, or for computations in the (very large) space of permutations (for instance in Twu and Egerstedt (2010) it is shown that certain problems are NP-hard in the indexed setting). The key observation is that the quotient space $\mathbb{R}^N / \sim$ is nothing more than the set of finite subsets of $\mathbb{R}^n$ of cardinality N. In other words, the joint state space of the multiagent system is really a subset of $2^{\mathbb{R}^n}$, and we can represent the state by an indicator function (or more appropriately, an indicator *distribution*) over $\mathbb{R}^n$. Dynamics then become partial differential equations (in particular, the advection equation) which evolve the indicator distribution. Using the language of partial differential equations (and in particular those describing fluids), this can be thought of as

an *Eulerian* view of the problem, whereas the traditional view in the multiagent systems literature is *Lagrangian*.

Similarly-motivated work includes Kloder and Hutchinson (2005) and Kloder et al. (2004), in which the individual configurations in $\mathbb{R}^2 \cong \mathbb{C}$ of robots in a formation are represented by the roots of a complex polynomial whose coefficients constitute the permutation-invariant joint configuration. Particularly attractive properties of this representation include that it is finite-dimensional and can be interpolated in a very straightforward way. A limitation not shared by our work is that it is only applicable to states/configurations in two dimensions, and since the representation hinges on the Fundamental Theorem of Algebra it is not clear how one would generalize to higher (or lower) dimensions.

Other work which is technically similar to ours also exists, mostly in the mathematical biology community under the heading of nonlocal integro-differential models of swarms; this includes Mogilner and Edelstein-Keshet (1999), Topaz et al. (2006), Laurent (2007), Bodnar and Velazquez (2005), and Burger et al. (2007). The models studied in this area typically also include diffusion and sometimes nonlinear reaction terms, but share the property of determining advection velocities by a convolution integral; hence many of the results given in e.g. Topaz et al. (2006) can be specialized to the case which we study. Nevertheless, there are a number of essential novelties to our approach which arise precisely because of the interplay between the indicator distribution and the indexed representation (the latter of which is not present in a purely continuum model). Among these are a natural permutation-invariant geometric structure which the indexed representation inherits via the *kernel trick*, and an Eulerian control philosophy which in certain situations can result in dramatically simple controllers.

The body of this paper proceeds as follows: After introducing the indicator distribution and its dynamics (section 2), we relate this representation to the more standard graph-theoretic one (section 3); then we prove conservation properties including stability (section 3.1). Following this we

construct an inner product space on indicator distributions which enables us to reason geometrically about them (and kernelize the Lagrangian representation) (section 4), and finally use a discrete analogue to this representation to give an example both of the dual relationship between the Eulerian and Lagrangian approaches, and of the utility for certain problems of selecting the Eulerian approach (section 5). In the interest of clean expressions, we freely drop arguments to functions throughout this document wherever this should not cause undue confusion.

2. THE INDICATOR DISTRIBUTION

Beginning from the classical notion of an indexed set of agents, we will build an indicator distribution, or permutation-invariant state, that retains all necessary information while stripping out agent identities. The essential idea will be to construct an object that tells us not what state each agent is in, but rather how many agents are in each state. The construction is as follows: Starting from an indexed collection of agents with states $x_1, \cdots, x_N \in \mathbb{R}^n$, we build an indicator distribution m over $\mathbb{R}^n$ defined,

$$m(x) = \Phi(x_1, \cdots, x_N)(x) = \sum_{i=1}^{N} \delta(x - x_i) \qquad (2)$$

where δ is the Dirac delta distribution on $\mathbb{R}^n$, $\mathcal{T}(\mathbb{R}^n)$ denotes the space of tempered distributions on $\mathbb{R}^n$, and the map $\Phi : \mathbb{R}^{nN} \to \mathcal{T}(\mathbb{R}^n)$ creates m from $x_1, \cdots, x_N$. We use m to denote this distribution because we would like to think of it as the agent "mass distribution." Importantly, notice that although indices were used in the construction of m, it is fundamentally an object that is concerned only with the number of agents in any given state.

One may also think of m as the probability distribution (after normalizing by $\frac{1}{N}$) that answers the question, "If an agent is chosen uniformly at random, what is the probability that that agent is at the state x?"

More generally, we would like to be able to add and subtract distributions so that we have a full vector space structure. Hence we will also consider linear combinations of distributions of the form (2); these take the form

$$x \mapsto \sum_{i=1}^{K} c_i \delta(x - \xi_i) \qquad (3)$$

for some $c_1, \cdots, c_K \in \mathbb{R}$, $K \in \mathbb{N}$, and $\xi_1, \cdots, \xi_K \in \mathbb{R}^n$.

Now suppose that, in the classical setting, each agent $i \in \{1, \cdots, N\}$ has state $x_i \in \mathbb{R}^n$ and dynamics

$$\dot{x}_i = v_i \qquad (4)$$

where v_i is our control input, and that moreover v_i is output by some controller, identical for all agents, that depends only on $m(\cdot, t)$ and not the indexed set of states. In other words (and making time dependence explicit), $v_i(t) = v(x_i(t), m(\cdot, t)) \; \forall i \in \{1, \cdots N\}$. Then, the equivalent dynamics for our indicator distribution are given by the *advection equation*

$$\begin{aligned} \dot{m} &= -\operatorname{div}(mv) \\ &= -\nabla \cdot (mv) \\ &= -(\nabla m) \cdot v - m(\nabla \cdot v) . \end{aligned} \qquad (5)$$

3. WEIGHTED LINEAR CONSENSUS

In the indexed setting, a particular problem which has received a great deal of attention is that of *distributed averaging*. This is described by the *consensus equation*

$$\dot{x}(t) = -L_w(\mathcal{G}(t))x(t) \qquad (6)$$

where $L_w(\mathcal{G}(t))$ is the (possibly weighted) *graph Laplacian* for some undirected interaction graph $\mathcal{G}(t)$ on N vertices, and can be written as the product

$$L_w(\mathcal{G}(t)) = D(t)W(t)D^T(t) \qquad (7)$$

where $D(t)$ is the *incidence matrix* for any orientation of $\mathcal{G}(t)$ and $W(t)$ is a diagonal matrix of positive edge weights. See Mesbahi and Egerstedt (2010) for an overview.

We say that the protocol (6) is *permutation invariant* if both the presence of an edge between two agents, and the weight assigned to an edge, are functions only of the permutation-invariant state (or indicator distribution) $m = \Phi(x)$. In other words, the interaction graph is allowed to depend on the states of the many agents, but not on their identities.

This occurs in a great many cases of interest, including disk graphs, Gabriel and Delaunay graphs, nearest-neighbor graphs, and even situations in which edges are functions of many agents' states (e.g., if a line-of-sight communication link between two agents can be severed by a third agent who gets in the way). Consensus on static (i.e., constant) interaction graphs, however, is generally not permutation-invariant, since in this case edges are determined not by agents' states but by their identities (i.e., if agent i is to communicate with agent j and not with agent k regardless of the many agents' states, then it requires some way to differentiate between agents j and k). In graph-theoretic terms, what is required for a static interaction graph is that every permutation of the vertex labels be a graph automorphism – a property possessed only by the empty graph and the complete graph. Hence, permutation-invariance will usually involve dynamic, state-dependent graphs.

So long as (6) is permutation-invariant, it is possible to express equivalent dynamics in our index-free framework. In short, the state-dependent vector field v in (5) then takes the form,

$$v = \int_{\zeta \in \mathbb{R}^n} m(\zeta)w(\zeta, x, m)(\zeta - x)d\zeta \qquad (8)$$

where $w : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}(\mathbb{R}^n) \to [0, \infty)$ is a positive state-dependent weighting function satisfying the symmetry property $w(\zeta, x, m) = w(x, \zeta, m)$ for all $x, \zeta \in \mathbb{R}^n$ and $m \in \mathcal{T}(\mathbb{R}^n)$. In most cases that arise, w depends only on its first two arguments, since the intensity with which a pair of agents interacts is usually a function only of their two states and not on other agents'. For instance, the unit-disk topology [1] can be encoded by the weighting function

$$w(\zeta, x) = \operatorname{ind}(||\zeta - x|| < 1) \qquad (9)$$

where ind([expression]) denotes the indicator function for the set where [expression] evaluates to 'true.'

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[1] I.e., two agents can interact if and only if they are within one unit distance of one another.

A yet-more-specific form for w which will be of particular interest is

$$w(\zeta, x) = f'\left(\frac{1}{2}||\zeta - x||^2\right) \qquad (10)$$

where f' is the derivative of some nondecreasing scalar function $f : \mathbb{R} \to \mathbb{R}$.

Regardless of the particular form of w, the controller (8) then induces the closed-loop dynamics on the indicator distribution m,

$$\dot{m}(x) = -\operatorname{div}\left(m(x)\int_{\zeta \in \mathbb{R}^n} m(\zeta)w(\zeta, x, m)(\zeta - x)d\zeta\right) . \qquad (11)$$

3.1 Properties of Index-Free Linear Consensus

In this section we prove center-of-mass conservation and stability properties of the closed-loop system (11). Note that the following theorems hold not only when m is a sum of Dirac deltas as in (2), but also for any positive m which either has compact support or which more generally vanishes at infinity; naturally, this includes smooth density functions.

We note here that similar results to Theorems 1 and 2 are also proven for a closely-related PDE featuring diffusive terms in Topaz et al. (2006). We nevertheless include Theorems 1 and 2 both in the interest of completeness, and for their value as instructive specializations to our case.

Theorem 1. (Center of Mass Conservation). Under the assumption that m vanishes at infinity, the center of mass $\bar{x}(t) \in \mathbb{R}^n$ defined by

$$\bar{x}^i \triangleq \langle m, x^i \rangle_{L_2(\mathbb{R}^n, \mathbb{R})} \qquad (12)$$

(where x^i is the ith canonical coordinate function) is constant in time. [2]

Proof : Differentiating the i-th component of $\bar{x}$ (and dropping the subscript $L_2(\mathbb{R}^n, \mathbb{R})$ to the inner product),

$$\frac{\partial}{\partial t} \langle m, x^i \rangle = \langle \dot{m}, x^i \rangle$$
$$= \langle -\operatorname{div}(mv), x^i \rangle$$
$$= -\sum_{j=1}^n \left\langle \frac{\partial}{\partial x^j}(mv)^j, x^i \right\rangle .$$

Since m vanishes at infinity, integration by parts gives that this equals

$$\left\langle (mv)^1, \frac{\partial}{\partial x^1}x^i \right\rangle + \cdots + \left\langle (mv)^n, \frac{\partial}{\partial x^n}x^i \right\rangle$$
$$= \left\langle (mv)^1, \delta^{1i} \right\rangle + \cdots + \left\langle (mv)^n, \delta^{ni} \right\rangle$$

where δ^{ij} denotes the Kronecker delta. As a result,

$$\frac{\partial}{\partial t} \langle m, x^i \rangle = \langle mv^i, \mathbf{1} \rangle$$

where $\mathbf{1}$ denotes the constant function that returns $1 \in \mathbb{R}$; this inner product is interpreted as the total mass flux. Expanding this expression, we have

$$\langle mv^i, \mathbf{1} \rangle = \int_{x \in \mathbb{R}^n} m(x) \int_{\zeta \in \mathbb{R}^n} m(\zeta)w(\zeta, x, m)$$
$$\cdot (\zeta - x)d\zeta dx$$
$$= \int_{(x,\zeta) \in \mathbb{R}^{2n}} m(x)m(\zeta)w(\zeta, x, m)$$
$$\cdot (\zeta - x)d(x, \zeta) .$$

We note that the term being integrated is antisymmetric in x and ζ, and hence that the integral is zero. ∎

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Theorem 2. (Stability). Let $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ take the form (10). Then the closed-loop dynamics (11)

- are *stable* if $f'(x) \geq 0 \ \forall x \in [0, \infty)$, and
- are *globally asymptotically stable* with equilibrium point $\bar{x}$ defined by (12), if $f'(x) > 0 \ \forall x \in [0, \infty)$

so long as m vanishes at infinity. Moreover, in the first case, if $f'(x) > 0 \ \forall x \in [0, R)$, then for all $i, j \in \{1, \cdots, N\}$, at equilibrium either $||x_i - x_j|| = 0$ or $||x_i - x_j|| > R$.

Proof : Consider the Lyapunov functional,

$$V(m) = \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} m(x)m(y)f\left(\frac{1}{2}||x - y||^2\right) dxdy . \qquad (13)$$

Differentiating,

$$\dot{V} = \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} [\dot{m}(x)m(y) + m(x)\dot{m}(y)]$$
$$\cdot f\left(\frac{1}{2}||x - y||^2\right) dxdy$$

which by symmetry

$$= 2\int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} \dot{m}(x)m(y)f\left(\frac{1}{2}||x - y||^2\right) dxdy$$

and substituting in (5)

$$= -2\int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} \operatorname{div}_x(m(x)v(x))m(y)$$
$$\cdot f\left(\frac{1}{2}||x - y||^2\right) dxdy$$
$$= -2\sum_{i=1}^n \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} \frac{\partial}{\partial x^i}(m(x)v^i(x))m(y)$$
$$\cdot f\left(\frac{1}{2}||x - y||^2\right) dxdy .$$

Integrating by parts (and since m vanishes at infinity), this is,

$$-2\sum_{i=1}^n \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} (m(x)v^i(x))$$
$$\cdot \frac{\partial}{\partial x^i}\left(m(y)f\left(\frac{1}{2}||x - y||^2\right)\right) dxdy$$

which, since $f'(\frac{1}{2}||x - y||^2) = w(x, y)$, simplifies to

$$2\sum_{i=1}^n \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} m(x)v^i(x)m(y)w(x, y)(x^i - y^i)dxdy .$$

Substituting in (8) we arrive at the separable integral

$$2\sum_{i=1}^{n}\int_{x\in\mathbb{R}^n} m(x)\left(\int_{y\in\mathbb{R}^n} m(y)w(x,y)(x^i - y^i)dy\right)$$
$$\cdot\left(\int_{\zeta in\mathbb{R}^n} m(\zeta)w(x,\zeta)(\zeta^i - x^i)d\zeta\right)dx$$
$$=-2\sum_{i=1}^{n}\int_{x\in\mathbb{R}^n} m(x)||v(x)||^2 dx$$

which is never positive so long as $f' \geq 0$ (thus proving the first statement of the theorem), and always strictly negative provided $f' > 0$ (which, together with the previous theorem, proves the second statement). The third statement likewise follows immediately from LaSalle's invariance principle. ∎

4. AN INNER PRODUCT SPACE VIA SMOOTHING

In this section we imbue the state space of indicator distributions with an inner product structure; this will enable us to reason geometrically about multiagent control laws in an index-free way. Essentially, we will smooth indicator distributions by convolving them with an appropriate function (in particular, a Gaussian) to arrive at smooth functions for which the standard L_2 inner product is defined; this is illustrated by figure 1. The reason for choosing Gaussians in particular is that the corresponding convolution operator is invertible. In geometric language, the inner product is constructed as the *pullback* of the standard L_2 inner product under a linear isomorphism.

The smoothing operator is introduced in section 4.1, which is used to define the inner product in section 4.2. Finally, in section 4.3, we use this inner product together with the embedding Φ introduced in (2) to compute these inner products directly from the indexed representation.

4.1 Smoothed Indicator Distributions

Let $w : \mathbb{R}^n \to \mathbb{R}$ be a Gaussian of the form,
$$w(x) = \exp(-x^T Q x) \tag{14}$$
for some $Q = Q^T \succ 0 \in \mathbb{R}^{n\times n}$. Then we define the *smoothing operator* $A_w : \mathcal{T}(\mathbb{R}n) \to L_2(\mathbb{R}^n, \mathbb{R})$ by,
$$A_w(m) = w * m \tag{15}$$
where $*$ is the standard convolution operator.

Lemma 1. A_w is a linear isomorphism.

Proof : A_w is clearly linear. To show that it is also invertible, we note that $\tilde{m} = A_w(m)$ can be computed as the solution to Laplace's equation in the following way: The PDE
$$\dot{\phi}(x,t) = \Delta\phi(x,t) \tag{16}$$
with the initial conditions $\phi(z,0) = m(Q^{-1/2}z)$ (where $(Q^{1/2})^T Q^{1/2} = Q$; $Q^{1/2}$ exists since $Q \succ 0$, and can be computed by e.g. the Cholesky decomposition) has the solution
$$\phi(z,t) = \sum_{i=1}^{N}\frac{1}{(4\pi t)^{n/2}}e^{-\frac{(z-Q^{1/2}x_i)^T(z-Q^{1/2}x_i)}{4t}} \tag{17}$$
for $t > 0$, and hence $\tilde{m}(x) = \pi^{n/2}\phi(Q^{1/2}x, \frac{1}{4})$. In the same way, starting with $\tilde{m}$ and imposing the condition

Indicator Distribution, m
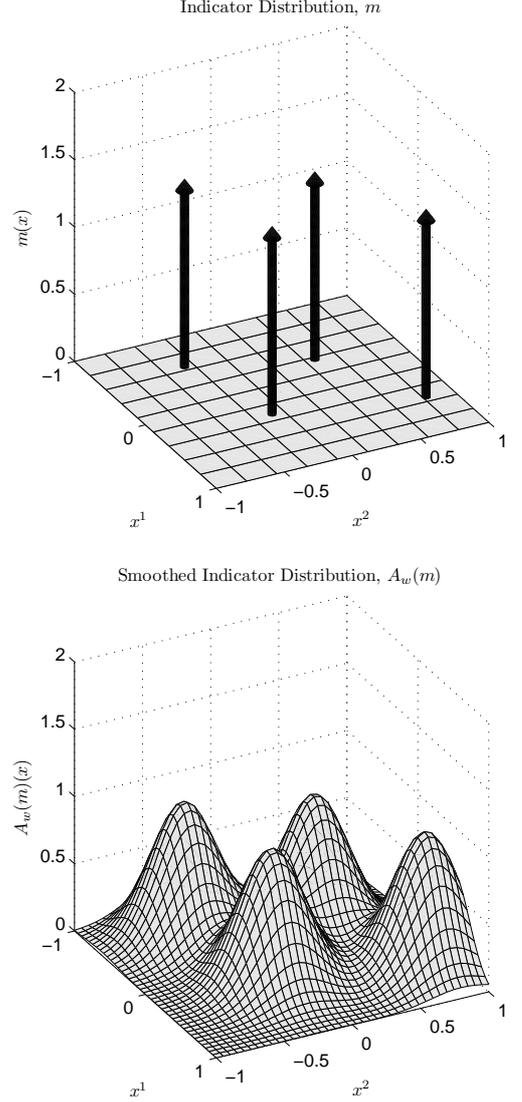

Smoothed Indicator Distribution, $A_w(m)$

Fig. 1. The indicator distribution (top) is smoothed by convolution with a Gaussian to arrive at a function in L_2 (bottom).

$\phi(z, \frac{1}{4}) = \frac{1}{\pi^{n/2}}\tilde{m}(Q^{-1/2}z)$, m can be reconstructed as $m(x) = \phi(Q^{1/2}x, 0)$. ∎

Note that although the backwards heat equation used in lemma 1 is extremely ill-conditioned, in principle the solution exists; see e.g. Evans (1998).

Lemma 2. For any indicator distribution m, $A(m)$ is square integrable.

Proof : $A(m)$ can be written as a sum of translated copies of w; since w is square-integrable, this sum is square-integrable. ∎

4.2 An Inner Product

Given two indicator distributions m_1, m_2, we define their inner product
$$\langle m_1, m_2 \rangle_A = \langle A(m_1), A(m_2) \rangle_{L_2(\mathbb{R}^n, \mathbb{R})} .$$

Lemma 3. $\langle \cdot, \cdot \rangle_A$ is an inner product.

Proof : Since $A(m_1)$ and $A(m_2)$ are square integrable by Lemma 2, their L_2 inner product exists. Since $\langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^n, \mathbb{R})}$ is symmetric, so is $\langle \cdot, \cdot \rangle_A$; since A is linear and $\langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^n, \mathbb{R})}$ is bilinear, $\langle \cdot, \cdot \rangle_A$ is bilinear; and since $\langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^n, \mathbb{R})}$ is positive definite and A is an isomorphism, $\langle \cdot, \cdot \rangle_A$ is positive definite. Hence $\langle \cdot, \cdot \rangle_A$ is an inner product. ∎

4.3 Kernelizing the Inner Product

How does the inner product of the previous section relate to the classical indexed representation of a multiagent system?

The *kernel* κ_Φ attached to the embedding Φ is the map

$$\kappa((x_1^1, \cdots, x_N^1), (x_1^2, \cdots, x_N^2)) \triangleq$$
$$\langle \Phi(x_1^1, \cdots, x_N^1), \Phi(x_1^2, \cdots, x_N^2) \rangle . \quad (18)$$

In other words, it is the map that computes inner products between joint states in the higher-dimensional space of indicator distributions, without (necessarily) needing to explicitly construct the indicator distribution representation (Aizerman et al. (1964) has more on the subject of kernel functions).

For indicator distributions of the form (2) and the inner product $\langle \cdot, \cdot \rangle_A$ defined in the previous section, the corresponding kernel is,

$$\kappa((x_1^1, \cdots, x_N^1), (x_1^2, \cdots, x_N^2))$$
$$= \int_{x \in \mathbb{R}^n} \prod_{i=1}^{2} \sum_{j=1}^{N} A_w(\delta)(x - x_j^i)$$
$$= (A_w(\delta)(x - x_1^2) + \cdots + A_w(\delta)(x - x_N^2)) \, dx$$
$$= \sum_{i,j} \int_{x \in \mathbb{R}^n} A_w(\delta)(x - x_i^1) A_w(\delta)(x - x_j^2) dx . \quad (19)$$

For the case of Gaussian κ as in (14),

$$\int_{x \in \mathbb{R}^n} A_w(\delta)(x - x^1) A_w(\delta)(x - x^2) dx$$
$$= \int_{x \in \mathbb{R}^n} \exp(-(x - x^1)^T Q(x - x^1))$$
$$\cdot \exp(-(x - x^2)^T Q(x - x^2)) dx$$
$$= \exp(-\frac{1}{2}(x^2 - x^1)^T Q(x^2 - x^1))$$
$$\cdot \int_{x \in \mathbb{R}^n} \exp(-(x - \bar{x}^{12})^T Q(x - \bar{x}^{12})) dx$$
$$= \exp(-\frac{1}{2}(x^2 - x^1)^T Q(x^2 - x^1)) \sqrt{\frac{\pi^n}{\det Q}} \quad (20)$$

for any $x^1, x^2 \in \mathbb{R}^n$ (where $\bar{x}^{12} = (x^1 + x^2)/2$), and so

$$\kappa((x_1^1, \cdots, x_N^1), (x_1^2, \cdots, x_N^2))$$
$$= \sqrt{\frac{\pi^n}{\det Q}} \sum_{i,j} \exp(-\frac{1}{2}(x_j^2 - x_i^1)^T Q(x_j^2 - x_i^1)) . \quad (21)$$

More generally, we may be interested in allowing indicator distributions of the form (3). In this case we have another embedding Φ_2 which maps (3), thought of as a *formal sum*, to the distribution encoded by the same, and its corresponding kernel is

$$\kappa_{\Phi_2}(c_1^1 \delta(x - \xi_1^1) + \cdots + c_K^1 \delta(x - \xi_{K^1}^1),$$
$$c_1^2 \delta(x - \xi_1^2) + \cdots + c_K^2 \delta(x - \xi_{K^2}^2))$$
$$= \sqrt{\frac{\pi^n}{\det Q}} \sum_{i,j} c_i^1 c_j^2 \exp(-\frac{1}{2}(x_j^2 - x_i^1)^T Q(x_j^2 - x_i^1)) \quad (22)$$

which differs from (21) by the factors c_i^1, c_j^2 which are now included.

The significance of the kernel function we have obtained is that it gives the original indexed representation a permutation-invariant geometry which can be used to reason about multiagent control laws without necessarily needing to work at the level of partial differential equations. Moreover this geometry can be understood concretely in terms of indicator distributions, and this ties the Eulerian and Lagrangian approaches together.

5. A FINITE-STATE-SPACE ANALOGUE

One interpretation of the Eulerian view of multiagent systems which we have presented so far is that, rather than thinking of agents as making decisions about which actions to take, one can instead view the states themselves as making decisions about how many agents should be entering or leaving them – subject to the dynamical constraints imposed by the number of agents in each state. We may take this interpretation very literally, and consider situations in which, e.g., rooms of a building decide which robots should enter them at any time, or in which sectors of a warzone command various autonomous support vehicles to enter or leave them in response to changing demands or in order to meet an objective.

In short, what we are considering is "dumb robots in a smart environment."

A qualitative observation which motivates this is the complementary behavior of the mass distribution m of the agents (the Eulerian setting) to that of their joint state vector x (the Lagrangian setting). We observe that in the Eulerian setting, consensus corresponds to a very "peaky" distribution, in which all mass is concentrated at one point, whereas in the Lagrangian setting, consensus corresponds to a "flat" distribution of states over agents, in which $x_1 = x_2 = \cdots = x_N$. Hence we can expect, more generally, that each controller in the Lagrangian setting corresponds to a "dual" controller in the Eulerian setting, and that certain control objectives may be easier to achieve in one setting or in the other; this is the idea explored in section 5.1.

The finite state spaces of these examples – e.g. rooms of a building – also motivate the construction of a finite-state-space analogue to the indicator-distribution representation we have discussed so far. To this end, we assume the existence of a set $R = \{1, \cdots, N\}$ of rooms, connected in an undirected graph $\mathcal{G}_p = (R, E_p \subset R \times R)$, in which edges indicate physical paths – e.g., hallways – by which agents can move between them. We likewise assume that the rooms can communicate via some network, represented as another graph $\mathcal{G}_c = (R, E_c \subset R \times R)$. Finally, we associate to each room $i \in R$ a number $m_i \in \mathbb{R}$ of agents currently in that room, and thereby define the vector $m = (m_1, \cdots, m_N) \in \mathbb{R}^N$ [The relaxation to allow for

a real (rather than only natural) number of agents in each room can be viewed as a limiting case for a very large number of agents]. Assuming either a discrete timestep or Lebesgue sampling, the dynamics of the resulting system are summarized,

$$m[k+1] = m[k] + Du[k] \quad \forall k \in \mathbb{N} \qquad (23)$$

subject to the elementwise state constraint $m[k] \geq 0$ for all $k \in \mathbb{N}$, where D is the incidence matrix associated with $\mathcal{G}_p$ and $u[k] \in \mathbb{R}^{|E_p|}$.

5.1 Example: Vacuuming an Office Building

As a particular concrete example, suppose that each room of an office building has been outfitted with a short-range (e.g., Bluetooth) wireless access point, each with some computing capacity that we can use, and that we have a number of vacuum-cleaning robots which we would like the access points to deterministically direct through the building to achieve a uniform distribution over the rooms, thereby minimizing the amount of dirt left in the worst-cleaned room. The question then becomes how the rooms of the building, in a distributed way and while respecting state constraints, can choose to direct robots between themselves so that they are eventually distributed uniformly throughout the building. This is a version of the *coverage problem*, and it will turn out to be particularly easy to solve in the Eulerian setting.

For the purposes of this example we will assume $\mathcal{G}_p = \mathcal{G}_c = \mathcal{G}$ – i.e., that the physical and network topologies are the same – in which case the controller

$$u[k] = -\gamma D^T m[k] \qquad (24)$$

for some $\gamma > 0$ gives the closed-loop dynamics

$$\begin{aligned} m[k+1] &= (I - \gamma DD^T)m[k] \\ &= (I - \gamma L)m[k] \qquad (25) \end{aligned}$$

where L is the graph Laplacian for $\mathcal{G}$. Since L is positive semidefinite (see Mesbahi and Egerstedt (2010)), the eigenvalues of the closed-loop system matrix lie within the unit circle for sufficiently small γ (for all but the $\mathbf{1}$ eigenvector, whose eigenvalue is exactly 1), e.g. $\gamma = \frac{1}{2}||L||_2$, and so the system is stable. Moreover, it can be seen that the given controller satisfies the state constraint $m[k] > 0 \quad \forall k \in \mathbb{N}$.

6. CONCLUSIONS

We have stripped agent identities from the multiagent modeling machinery by employing an indicator function representation, and in so doing arrived at an integro-differential model for multiagent systems which parallels the now-standard graph-theoretic constructions. Along the way, we proved stability and conservation properties from within a continuum model, and, guided by our permutation-invariant representation and the so-called kernel trick, endowed the traditional vector state space with a permutation-invariant geometry. Finally, we illustrated a qualitative duality between the Eulerian and Lagrangian approaches by way of a finite-state-space analogue, which demonstrated that for certain problems a literal interpretation of the Eulerian approach can result in very simple controllers.

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